Chapter 1 — Exponents and logarithms

Conversion between exponents and logarithms

The conversion between exponents and logarithms is defined below:

For $a > 0$, $a^x = b \Leftrightarrow x = \log_a b$.

When $a = 10$, we express the operator $\log_{10}$ as $\log$ and this can be found on the GDC.

e is a special constant with numerical value $2.71828\ldots$ and can be found on the GDC.

When $a = e$, we can express the operator $\log_e$ as $\ln$ and this can be found on the GDC also.

Example: by using the above notation, we have the following equivalent equations:

$4^3 = 64 \Leftrightarrow 3 = \log_4 64$

$10^4 = 10000 \Leftrightarrow 4 = \log 10000$

$e^2 = 7.39$ (3sf) $\Leftrightarrow 2 = \ln 7.39$

Laws of exponents

The following table lists the laws of exponents (m and n are real numbers while a and b are non-zero real numbers unless specified otherwise.)

Laws	Examples
$a^m a^n = a^{m+n}$	$(3^4)(3^2) = 3^{4+2} = 3^6$
$\frac{a^m}{a^n} = a^{m-n}$	$\frac{2^7}{2^3} = 2^{7-3} = 2^4$
$(a^m)^n = a^{mn}$	$(5^3)^4 = 5^{3\times4} = 5^{12}$
$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m,\ n \neq 0$	$16^{\frac{3}{2}} = \left(\sqrt{16}\right)^3 = 4^3 = 64,\ 27^{\frac{5}{3}} = \left(\sqrt[3]{27}\right)^5 = 3^5 = 243$
$(ab)^n = a^n b^n$	$(2x)^7 = 2^7 x^7 = 128x^7$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{k}{3}\right)^4 = \frac{k^4}{3^4} = \frac{k^4}{81}$
$a^{-n} = \frac{1}{a^n}$	$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$
$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\left(\frac{7}{9}\right)^{-3} = \left(\frac{9}{7}\right)^3 = \frac{9^3}{7^3}$
$a^0 = 1,\ a \neq 0$	$68^0 = 1$
$0^n = 0,\ n > 0$	$0^{295} = 0$

There are two points to note here:

(i) 0 to the power of a negative number does not work because, for instance, $0^{-5} = \frac{1}{0^5} = \frac{1}{0}$ and we cannot divide any number by 0.

(ii) A negative number to an even power will get a positive answer, but a negative number to an odd power will get a negative answer. There is another way to remember this:

$$\begin{cases} \text{even power of a non-zero number is always positive} \\ \text{odd power of a non-zero number is always the same sign as before} \end{cases}$$

Laws of logarithms

Laws	Examples
$\log_b x + \log_b y = \log_b xy$	$\log_7 2 + \log_7 5 = \log_7(2(5)) = \log_7 10$
$\log_b x - \log_b y = \log_b\left(\frac{x}{y}\right)$	$\log_8 13 - \log_8 4 = \log_8\left(\frac{13}{4}\right)$
$\log_b(x^p) = p\log_b x$	$\log_{11}(6^5) = 5\log_{11} 6$
$\log_a a = 1$	$\log_5 5 = 1$
$\log_a 1 = 0$	$\log_{31} 1 = 0$
$a^x = e^{x\ln a}$	$9^7 = e^{7\ln 9}$
$\log_a a^x = x = a^{\log_a x}$	$\log_6 6^8 = 8 = 6^{\log_6 8}$
$\log_b a = \frac{\log_c a}{\log_c b}$	$\log_4 11 = \frac{\log_3 11}{\log_3 4}$

The last one is called the change of base formula.

To input a logarithmic function into the GDC, we use the change of base formula to convert the base of the logarithmic function to either 10 or e so that we can use $\log$ or $\ln$ in the GDC. Therefore, we have $\log_b a = \frac{\log_{10} a}{\log_{10} b} = \frac{\log a}{\log b}$ and $\log_b a = \frac{\log_e a}{\log_e b} = \frac{\ln a}{\ln b}$.

Note that $\log_a x$ is undefined if $x \le 0$. So $\log_2(0)$, $\log_4(-3)$ and $\ln(-2.5)$ are undefined. We'll investigate more in chapter 8.

Example: for $\log_3 72$, we can express it in terms of $\log_3 2$ as follows:

$\log_3 72$

$= \log_3(2^3 \times 3^2)$

$= \log_3 2^3 + \log_3 3^2$

$= 3\log_3 2 + 2\log_3 3$

$= 3\log_3 2 + 2(1)$

$= 3\log_3 2 + 2$

Example: to find the solution of the equation $7^x = 19$ to 3 significant figures, we need to make use of the change of base formula and the GDC:

$7^x = 19$

$\Rightarrow x = \log_7 19 = \dfrac{\log 19}{\log 7} = 1.51$ (3sf)

Worked examples

Example question 1.1: (HL/SL Paper 1 Section A)

Find the lowest common multiple of $30x^2y^3z^5$ and $36x^3yz^4$.

Solution:

Since we can express 30 as $2(3)(5)$ and 36 as $2^2 3^2$, the lowest common multiple of $30x^2y^3z^5$and $36x^3yz^4$ is $2^2 3^2 5x^3y^3z^5 = 180x^3y^3z^5$.

Example question 1.2: (HL/SL Paper 1 Section A)

Solve $\log_3\left((x-3)^2\right) = 4$ for x.

Solution:

$\log_3(x-3)^2 = 4$

$\Rightarrow (x-3)^2 = 3^4$

$\Rightarrow (x-3)^2 = 81$

$\Rightarrow x - 3 = \pm 9$

$\Rightarrow x = 12$ or -6

Example question 1.3: (HL/SL Paper 2 Section A)

Solve $(7x)^{-\frac{3}{2}}\left(\frac{14}{3}\right)\left(\frac{12}{x^2}\right)^2 = 4x^{-5}$.

Solution:

$$(7x)^{-\frac{3}{2}}\left(\frac{14}{3}\right)\left(\frac{12}{x^2}\right)^2 = 4x^{-5}$$

$$\Rightarrow \frac{1}{7^{\frac{3}{2}}x^{\frac{3}{2}}}\left(\frac{2(7)}{3}\right)\left(\frac{144}{x^4}\right) = \frac{4}{x^5}$$

$$\Rightarrow \frac{24}{7^{\frac{1}{2}}} = x^{\frac{1}{2}}$$

$$\Rightarrow x = \frac{576}{7}$$

Example question 1.4: (HL/SL Paper 2 Section A)

Solve $\log_x 13 + 2 = \ln 17$.

Solution:

$$\log_x 13 + 2 = \ln 17$$

$$\Rightarrow \log_x 13 = \ln 17 - 2$$

$$\Rightarrow \frac{\log 13}{\log x} = \ln 17 - 2$$

$$\Rightarrow \log x = \frac{\log 13}{\ln 17 - 2}$$

$$\Rightarrow x = 10^{\left(\frac{\log 13}{\ln 17 - 2}\right)} = 21.7 \text{ (3sf)}$$

Exercises

Exercise 1.1 (HL/SL Paper 1 Section A)

Simplify $\left(\frac{4}{9}x^2\right)^{-\frac{1}{2}}\left(\frac{-\frac{7}{3}}{(27x)^{-\frac{1}{3}}}\right)$ and give the answer in terms of a positive power of x.

Answer: $-\frac{21}{2x^{\frac{2}{3}}}$

Exercise 1.2 (HL/SL Paper 1 Section A)

Express $\log_2\left(\frac{25}{16}\right)$ in terms of k where $k = \log_2 5$.

Answer: $2k - 4$

Exercise 1.3 (HL/SL Paper 2 Section A)

Find the highest common factor of $56875a^2b^5c^7$ and $44200a^{10}b^4c^3$.

Answer: $325a^2b^4c^3$

Exercise 1.4 (HL/SL Paper 2 Section A)

Solve $2(13^x) = (5e)^{-x}$. Give the answer to 3 significant figures.

Answer: -0.134

Chapter 18 — Introduction to complex numbers*

Complex numbers

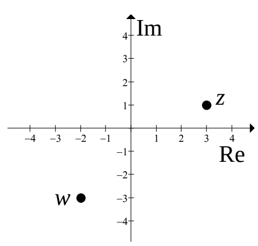
The square root of -1 is defined as $i = \sqrt{-1}$. Therefore, we also have $i^2 = -1$.
Example: $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$
$\sqrt{-72} = \sqrt{36}\sqrt{2}\sqrt{-1} = 6\sqrt{2}\,i$

A complex number is usually expressed as the Cartesian form $z = a + bi$ where both a and b are the real numbers. a is called the real part while b is called the imaginary part. The notation is as follows:
$\mathrm{Re}(z) = \mathrm{Re}(a + bi) = a$, $\mathrm{Im}(z) = \mathrm{Im}(a + bi) = b$
Example: if $z = 7 - 3i$, then $\mathrm{Re}(z) = \mathrm{Re}(7 - 3i) = 7$ and $\mathrm{Im}(z) = \mathrm{Im}(7 - 3i) = -3$
Note that when $b = 0$, $z = a + 0i = a$, which is a real number, and when $a = 0$, $z = 0 + bi = bi$ which is called a pure imaginary number.

The conjugate of z is defined to be z^*(or $\bar{z}$) $= a - bi$.
Example: if $z = -3 + 6i$, then its conjugate is $z^* = -3 - 6i$.

The Argand diagram (the complex plane)

In the Cartesian coordinate plane, if we take the x-axis and y-axis as the real and imaginary axes, then it is called the Argand diagram (the complex plane) and it is used to represent the complex numbers graphically.
Example: if $z = 3 + i$ and $w = -2 - 3i$, then we can represent them graphically as follows:

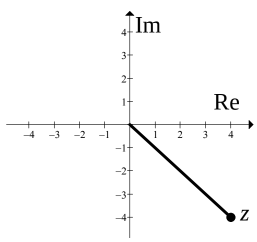


Modulus-argument form (Polar form)

When $z = a + bi$, the modulus and argument (principal argument) of z are defined as: modulus of $z = |z| = \sqrt{a^2 + b^2}$ and argument of $z = \arg(z) = \arctan\left(\frac{b}{a}\right)$

We may interpret the meanings of the modulus and argument in this way: if we draw a line from the origin to the point represented by the complex number on the Argand diagram, the modulus of the complex number is the length of this line segment while the argument is the angle between it and the positive x-axis. For the argument, θ, the range is taken to be $-\pi < \theta \leq \pi$ (in radians) or $-180^\circ < \theta \leq 180^\circ$ (in degree).

Example: if $z = 4 - 4i$, then $|z| = r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$ and $\arg(z) = \theta = \arctan\left(\frac{-4}{4}\right) = \arctan(-1)$. A sketch will help us to determine the argument:



Since the point represented by z is located in the fourth quadrant, this gives us: $\arg(z) = -\frac{\pi}{4}$ or -45°.

The modulus-argument form of a complex number z is given by $z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta = re^{i\theta}$ where r and θ are its modulus and argument respectively.

Example: we have shown in above that the modulus and argument of $4 - 4i$ are $4\sqrt{2}$ and $-\frac{\pi}{4}$ respectively, so its modulus-argument form is $4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$ or $4\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right) = 4\sqrt{2}\,e^{-\frac{\pi}{4}}$.

Four operations of complex numbers

The addition and subtraction of complex numbers are given by

$(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$. It's just like doing "normal" algebra.

Example: $(7+2i)+(4-8i) = 11-6i$

$(-4-5i)-(-7+12i) = 3-17i$

The multiplication of complex numbers is like doing "normal" algebra also, except we need to take $i^2 = -1$ into consideration in our calculation.

Example: $(4+3i)\times(5-8i)$

$= 20+15i-32i-24i^2$

$= 20-17i-24(-1)$

$= 44-17i$

The division of complex numbers is different from doing "normal" algebra. We need to "make the denominator real" by multiplying by its conjugate.

Example: $(10-4i)\div(3-2i)$

$= \dfrac{10-4i}{3-2i}$

$= \dfrac{(10-4i)(3+2i)}{(3-2i)(3+2i)}$

$= \dfrac{30-12i+20i-8i^2}{9-4i^2}$

$= \dfrac{30+8i-8(-1)}{9-4(-1)}$

$= \dfrac{38+8i}{13}$

$= \dfrac{38}{13}+\dfrac{8}{13}i$

Product and quotient of complex numbers in modulus-argument form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then their product would be

$z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i^2\sin\theta_1\sin\theta_2)$

$= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1))$

$= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

Therefore, we have $z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$.

For their quotient, we have:

$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$

$= \frac{r_1(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 - i\sin\theta_2\cos\theta_1 - i^2\sin\theta_1\sin\theta_2)}{r_2(\cos^2\theta_2 - i^2\sin^2\theta_2)}$

$= \frac{r_1(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1))}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$

$= \frac{r_1(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))}{r_2(1)}$

$= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$

Therefore, we have $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$.

Example: if $z_1 = 4\text{cis}\left(\frac{\pi}{6}\right)$ and $z_2 = 7\text{cis}\left(-\frac{\pi}{3}\right)$, then their products and quotients are:

$z_1z_2 = 4(7)\text{cis}\left(\frac{\pi}{6} + \left(-\frac{\pi}{3}\right)\right) = 28\text{cis}\left(-\frac{\pi}{6}\right)$;

$\frac{z_1}{z_2} = \frac{4}{7}\text{cis}\left(\frac{\pi}{6} - \left(-\frac{\pi}{3}\right)\right) = \frac{4}{7}\text{cis}\left(\frac{\pi}{2}\right)$ (which is $\frac{4}{7}i$, check!)

From the product and quotient of complex numbers in modulus-argument form above, we can deduce the following properties:

modulus: $|z_1z_2| = |z_1||z_2|$, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

argument: $\arg(z_1z_2) = \arg z_1 + \arg z_2$, $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Example: if $z_1 = 2\text{cis}\left(-\frac{\pi}{4}\right)$ and $z_2 = 8\text{cis}\left(\frac{\pi}{12}\right)$, then for $z_3 = z_1 z_2$ and $z_4 = \frac{z_1}{z_2}$, we have

$|z_3| = |z_1||z_2|$

$= 2(8) = 16$

$\arg z_3$

$= \arg z_1 + \arg z_2$

$= -\frac{\pi}{4} + \frac{\pi}{12}$

$= -\frac{\pi}{6}$

$|z_4| = \frac{|z_1|}{|z_2|}$

$= \frac{2}{8} = \frac{1}{4}$

$\arg z_4$

$= \arg z_1 - \arg z_2$

$= -\frac{\pi}{4} - \frac{\pi}{12}$

$= -\frac{\pi}{3}$

Worked examples

Example question 18.1: (HL Paper 1 Section A)

Find real numbers m and n such that $\frac{m}{1+i} - \frac{n}{2-i} = \frac{1-5i}{3-2i}$.

Solution:

First of all, we simplify the equation as follows:

$\frac{m}{1+i} - \frac{n}{2-i} = \frac{1-5i}{3-2i}$

$\Rightarrow \frac{m(2-i)-n(1+i)}{(1+i)(2-i)} = \frac{1-5i}{3-2i}$

$\Rightarrow \frac{2m-mi-n-ni}{2+2i-i-i^2} = \frac{1-5i}{3-2i}$

$\Rightarrow \frac{(2m-n)-(m+n)i}{3+i} = \frac{1-5i}{3-2i}$

$\Rightarrow [(2m-n)-(m+n)i](3-2i) = (3+i)(1-5i)$

$\Rightarrow (6m-3n)-(4m-2n)i-(3m+3n)i+(2m+2n)i^2 = 3+i-15i-5i^2$

$\Rightarrow (4m-5n)+(-7m-n)i = 8-14i$

By comparing the real and imaginary parts of both sides, we have $\begin{cases} 4m-5n = 8 \\ -7m-n = -14 \end{cases}$.

Multiplying the second equation by 5 and then subtracting it from the first equation gives

$4m-5n-5(-7m-n)$

$= 8-5(-14)$

$\Rightarrow 39m = 78$

$\Rightarrow m = 2$

Putting it back to the first equation, we get

$4(2)-5n = 8$

$\Rightarrow n = 0.$

Example question 18.2: (HL Paper 1 Section A)

Express $z = 1 - \sqrt{3}\,i$ in modulus-argument form.

Solution:

First of all, we need to identify that z is in quadrant IV in the Argand diagram. Then, the modulus and argument can be found as follows:

$$|z| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\arg(z) = \arctan\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$ (since z is in quadrant IV)

Therefore,

$$z = 1 - \sqrt{3}\,i$$

$$= 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$ or $2\text{cis}\left(-\frac{\pi}{3}\right)$ or $2e^{i(-\frac{\pi}{3})}$

Example question 18.3: (HL Paper 1 Section A)

Solve for z in the equation $\frac{1}{z} = 2 + 3i$. Express it in the form of $a + bi$.

Solution:

This is straightforward:

$$\frac{1}{z} = 2 + 3i$$

$$\Rightarrow z = \frac{1}{2 + 3i}$$

$$= \frac{1}{2 + 3i} \times \frac{2 - 3i}{2 - 3i}$$

$$= \frac{2 - 3i}{4 - 9i^2} = \frac{2 - 3i}{13}$$

$$= \frac{2}{13} - \frac{3}{13}i$$

Example question 18.4: (HL Paper 1 Section A)

Solve for z in the equation $\sqrt{z} = 4 - 6i$. Express the answer in the form of $a + bi$.

Solution:

This is straightforward:

$\sqrt{z} = 4 - 6i$

$\Rightarrow z = (4 - 6i)^2$

$= 16 - 48i + 36i^2$

$= -20 - 48i$

Example question 18.5: (HL Paper 1 Section B)

If z is a non-zero complex number and $w = \frac{z}{z^*}$, show that $\mathrm{Im}\left(\frac{2w}{1 + w^2}\right) = 0$.

Solution:

Let $z = a + bi$ where a and b are real numbers. We simplify the numerator and the denominator of $\frac{2w}{1 + w^2}$ respectively first before putting them together:

$2w$

$= \frac{2z}{z^*}$

$= \frac{2(a + bi)}{a - bi}$

$= \frac{2(a + bi)(a + bi)}{(a - bi)(a + bi)}$

$= \frac{2(a^2 + 2abi + b^2i^2)}{a^2 - b^2i^2}$

$= \frac{(2a^2 - 2b^2) + 4abi}{a^2 + b^2}$

$1 + w^2$

$= 1 + \left(\frac{z}{z^*}\right)^2$

$= 1 + \frac{(a + bi)^2}{(a - bi)^2}$

$= 1 + \frac{a^2 + 2abi + b^2i^2}{a^2 - 2abi + b^2i^2}$

$= 1 + \frac{(a^2 - b^2) + 2abi}{(a^2 - b^2) - 2abi}$

$$= \frac{(a^2 - b^2) - 2abi + (a^2 - b^2) + 2abi}{(a^2 - b^2) - 2abi}$$

$$= \frac{2a^2 - 2b^2}{(a^2 - b^2) - 2abi}$$

Putting them back together gives:

$$\frac{2w}{1 + w^2}$$

$$= \frac{\frac{(2a^2 - 2b^2) + 4abi}{a^2 + b^2}}{\frac{2a^2 - 2b^2}{(a^2 - b^2) - 2abi}}$$

$$= \frac{(2a^2 - 2b^2) + 4abi}{a^2 + b^2} \times \frac{(a^2 - b^2) - 2abi}{2a^2 - 2b^2}$$

$$= \frac{2((a^2 - b^2) + 2abi)((a^2 - b^2) - 2abi)}{(a^2 + b^2)(2a^2 - 2b^2)}$$

$$= \frac{2\left((a^2 - b^2)^2 - 4a^2b^2i^2\right)}{(a^2 + b^2)(2a^2 - 2b^2)}$$

$$= \frac{2\left((a^2 - b^2)^2 + 4a^2b^2\right)}{(a^2 + b^2)(2a^2 - 2b^2)}$$

We can stop here because we can see that the imaginary part is gone already, that is, it is zero. Therefore, we have shown that $\mathrm{Im}\left(\frac{2w}{1 + w^2}\right) = 0$.

Exercises

Exercise 18.1 (HL Paper 1 Section A)

Solve for z if $\frac{z - 6i}{z + 2i} = -i$.

Answer: $4 + 2i$

Exercise 18.2 (HL Paper 1 Section A)

Solve for z if $\frac{1}{z} + \frac{1}{1 - 0.5i} = 1$

Answer: $1 + 2i$

Exercise 18.3 (HL Paper 1 Section A)

Given that $z = \frac{2+3i}{5+i}$.

(a) Express z in the form of $a + bi$

(b) Find the modulus and argument of z.

Answer: (a) $\frac{1}{2} + \frac{1}{2}i$ (b) $\frac{1}{\sqrt{2}}$ (or $\frac{\sqrt{2}}{2}$), $\frac{\pi}{4}$

Exercise 18.4 (HL Paper 1 Section A)

Find the modulus of z if $|z+9| = 3|z+1|$ where z is a complex number.

Answer: 3

Exercise 18.5 (HL Paper 1 Section B)

Let $z = \sqrt{3} + i$.

(a) Find w if $w = \frac{z^*}{z}$.

(b) Find the modulus of w.

(c) Show that $\arg(w) = \arg(z^*) - \arg(z)$.

Answer: (a) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ (b) 1

Exercise 6.6 (HL Paper 1 Section B)

Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$